

A CLASS OF BANACH ALGEBRAS OF GENERALIZED MATRICES

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ABSTRACT. We introduce a class of Banach algebras of generalized matrices and study the existence of approximate units, ideal structure, and derivations of them.

1. INTRODUCTION

Let \mathfrak{X} be a compact metrizable space and \mathbf{m} be a Borel probability measure on \mathfrak{X} . In this note we study some aspects of the algebraic structure of a Banach algebra \mathcal{M} of generalized complex matrices whose their arrays are indexed by elements of \mathfrak{X}^2 and vary continuously. The multiplication of \mathcal{M} is defined similar to the ordinary matrix multiplication and uses \mathbf{m} as the weight for arrays. See Section 2 for exact definition. In the case that \mathbf{m} has full support, \mathcal{M} is isometric isomorphic to a subalgebra of compact operators acting on the Banach space of continuous functions on \mathfrak{X} . Indeed any element of \mathcal{M} defines an integral operator in a canonical way. Thus \mathcal{M} can be interpreted as a Banach algebra of integral operators or kernels ([2]). In Section 3 we investigate the existence of approximate units of \mathcal{M} . In Section 4 we show that if \mathfrak{X} is infinite then the center of \mathcal{M} is zero. In Section 5 we study ideal structure of \mathcal{M} . In Section 6 we consider some classes of representations of \mathcal{M} . In Section 7 we show that under some mild conditions bounded derivations on \mathcal{M} are approximately inner.

Notations. For a compact space \mathfrak{X} and a Banach space E we denote by $\mathbf{C}(\mathfrak{X}; E)$ the Banach space of continuous E -valued functions on \mathfrak{X} with supremum norm. We also let $\mathbf{C}(\mathfrak{X}) := \mathbf{C}(\mathfrak{X}; \mathbb{C})$. There is a canonical isometric isomorphism $\mathbf{C}(\mathfrak{X}; E) \cong \mathbf{C}(\mathfrak{X}) \hat{\otimes} E$ where $\hat{\otimes}$ denotes the completed injective tensor product. The phrase “point-wise convergence topology” is abbreviated to “pct”. By pct on $\mathbf{C}(\mathfrak{X}; E)$ we mean the vector topology under which a net $(f_\lambda)_\lambda \in \mathbf{C}(\mathfrak{X}; E)$ converges to f if and only if $f_\lambda(x) \rightarrow f(x)$ in the norm of E for every $x \in \mathfrak{X}$. If f and f' are complex functions on spaces \mathfrak{X} and \mathfrak{X}' then $f \otimes f'$ denotes the function on $\mathfrak{X} \times \mathfrak{X}'$ defined by $(x, x') \mapsto f(x)f'(x')$. The support of a Borel measure \mathbf{m} is denoted by Spm . $\mathcal{B}_{x,\delta}$ denotes the open ball with center at x and radius δ .

2. THE MAIN DEFINITIONS

Let \mathfrak{X} be a compact metrizable space and \mathbf{m} be a Borel probability measure on \mathfrak{X} . By analogy with matrix multiplication we let the convolution of $f, g \in \mathbf{C}(\mathfrak{X}^2)$ be defined by $f \star g(x, y) = \int_{\mathfrak{X}} f(x, z)g(z, y) d\mathbf{m}(z)$. Also by analogy with matrix adjoint we let $f^* \in \mathbf{C}(\mathfrak{X}^2)$ be defined by $f^*(x, y) = \overline{f(y, x)}$. It is easily verified that \star is an associative multiplication, $*$ is an involution, and also, $\|f \star g\|_\infty \leq \|f\|_\infty \|g\|_\infty$ and $\|f^*\|_\infty = \|f\|_\infty$; thus $\mathbf{C}(\mathfrak{X}^2)$ becomes

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a Banach $*$ -algebra which we denote by $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$. If \mathfrak{X} is a finite space with n distinct elements x_1, \dots, x_n and $\text{Spm} = \mathfrak{X}$ then the assignment $(a_{ij}) \mapsto ((x_i, x_j) \mapsto \frac{1}{\mathfrak{m}\{x_i\}} a_{ij})$ defines a $*$ -algebra isomorphism from the algebra of $n \times n$ matrices onto $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$.

Beside norm and pc topologies on $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ we need two other topologies: Consider the canonical isometric isomorphism $f \mapsto (y \mapsto f(\cdot, y))$ from $\mathbf{C}(\mathfrak{X}^2)$ onto $\mathbf{C}(\mathfrak{X}; \mathbf{C}(\mathfrak{X}))$. We define the column-wise convergence topology (cct for short) on $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ to be the pull back of the pct on $\mathbf{C}(\mathfrak{X}; \mathbf{C}(\mathfrak{X}))$ under this isomorphism. The row-wise convergence topology (rct for short) on $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ is defined similarly by using the other canonical isomorphism $f \mapsto (x \mapsto f(x, \cdot))$. The pct on $\mathbf{C}(\mathfrak{X}^2) = \mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ is contained in the intersection of cct and rct. Column-wise and row-wise cts are *adjoint* to each other in the sense that the involution $*$ from $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ with cct to $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ with rct is a homeomorphism. If $a_\lambda \xrightarrow{\text{cct}} a$ and $b_\lambda \xrightarrow{\text{rct}} b$ in $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$, then $c \star a_\lambda \xrightarrow{\text{cct}} c \star a$ and $b_\lambda \star c \xrightarrow{\text{rct}} b \star c$ for every c .

The assignment $(\mathfrak{X}, \mathfrak{m}) \mapsto \mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ can be considered as a cofunctor from the category of pairs $(\mathfrak{X}, \mathfrak{m})$ to the category of Banach $*$ -algebras: Suppose that $(\mathfrak{X}', \mathfrak{m}')$ is another pair of a compact metrizable space and a Borel probability measure on it. Let $\alpha : \mathfrak{X}' \rightarrow \mathfrak{X}$ be a measure preserving continuous map. Then α induces a bounded $*$ -algebra morphism $\mathcal{M}\alpha$ from $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ into $\mathcal{M}_{\mathfrak{X}', \mathfrak{m}'}$ defined by $[(\mathcal{M}\alpha)f](x', y') = f(\alpha(x'), \alpha(y'))$. By an explicit example we show that \mathcal{M} as a functor is not full: Let $\beta : \mathfrak{X} \rightarrow \mathbb{C}$ be a continuous function with $|\beta| = 1_{\mathfrak{X}}$ and $\beta \neq 1_{\mathfrak{X}}$. Then $\hat{\beta} : \mathcal{M}_{\mathfrak{X}, \mathfrak{m}} \rightarrow \mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ defined by $(\hat{\beta}f)(x, y) = \beta(x)f(x, y)\bar{\beta}(y)$ is an isometric $*$ -algebra isomorphism. It is clear that $\hat{\beta}$ is not of the form $\mathcal{M}\alpha$ for any $\alpha : \mathfrak{X} \rightarrow \mathfrak{X}$.

Let \mathfrak{X}_0 be a closed subset of \mathfrak{X} containing Spm and let $\iota : \mathfrak{X}_0 \rightarrow \mathfrak{X}$ denote the embedding. Then $\mathcal{M}\iota : \mathcal{M}_{\mathfrak{X}, \mathfrak{m}} \rightarrow \mathcal{M}_{\mathfrak{X}_0, \mathfrak{m}}$ is surjective with kernel $I := \{f : f|_{\mathfrak{X}_0^2} = 0\}$. Thus $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ is an extension of $\mathcal{M}_{\mathfrak{X}_0, \mathfrak{m}}$ by the closed self-adjoint ideal I . Moreover, suppose that \mathfrak{X}_0 is a retract of \mathfrak{X} i.e. there is a continuous map $\rho : \mathfrak{X} \rightarrow \mathfrak{X}_0$ with $\rho\iota = \text{id}_{\mathfrak{X}_0}$. It follows from functoriality of \mathcal{M} that $(\mathcal{M}\iota)(\mathcal{M}\rho)$ is the identity morphism on $\mathcal{M}_{\mathfrak{X}_0, \mathfrak{m}}$. This shows that the mentioned extension splits strongly in the sense of [1, Definition 1.2]. The discussion we just had, shows that by removing the null part of \mathfrak{m} from \mathfrak{X} we do not lose the principal part of the structure of $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$. We will see that $\text{Spm} = \mathfrak{X}$ is a crucial condition for the study of $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$.

For any closed subset C of \mathfrak{X} we have $\mathfrak{m}(C) = \inf[(1_{\mathfrak{X}} \otimes f) \star 1_{\mathfrak{X}^2}](x, y)$, the infimum being taken over all continuous functions f on \mathfrak{X} with $f(\mathfrak{X}) \subseteq [0, 1]$ and $f(C) = \{1\}$. Using this and inner regularity of \mathfrak{m} we can find the measure of any Borel subset. Hence we can recover \mathfrak{m} from $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$. The author does not know if the homeomorphism type of \mathfrak{X} can be recovered from $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$. Suppose that \mathfrak{X} is finite with $\text{Spm} = \mathfrak{X}$. It is not so hard to see that if $\phi : \mathcal{M}_{\mathfrak{X}, \mathfrak{m}} \rightarrow \mathcal{M}_{\mathfrak{X}', \mathfrak{m}'}$ is an isometric $*$ -isomorphism then there exist a measure preserving injective and surjective map $\alpha : \mathfrak{X}' \rightarrow \mathfrak{X}$ and a function $\beta : \mathfrak{X} \rightarrow \mathbb{C}$, with $|\beta| = 1_{\mathfrak{X}}$, such that $\phi = (\mathcal{M}\alpha)\hat{\beta}$, where $\hat{\beta}$ is defined as above. (Note that if ϕ is not supposed to be isometric then this assertion is wrong.) We suggest that this conclusion is true for any arbitrary \mathfrak{X} with $\text{Spm} = \mathfrak{X}$.

In Koopman's theory, as it is well known, the operator algebras have many applications to study of dynamical systems and ergodic theory ([3]). In this direction, the study of algebraic properties of $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ may be useful: Let G be a discrete group of measure preserving homeomorphisms of \mathfrak{X} . Then G acts on $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$ by isometric automorphisms and thus it is

appropriate to consider the crossed product Banach algebra $A := G \ltimes \mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$. It is clear that any algebraic invariant of A is an invariant of the dynamical system (\mathfrak{X}, G) . Moreover, if the suggestion stated in the preceding paragraph is true, then (\mathfrak{X}, G) is completely characterized by A . We plan to discuss elsewhere such possible connections with ergodic theory.

3. APPROXIMATE UNITS OF \mathcal{M}

From now on, \mathfrak{X} is a fixed compact metrizable space, \mathfrak{m} is a fixed Borel probability measure on \mathfrak{X} with $\text{Spm} = \mathfrak{X}$, and \mathcal{M} will denote $\mathcal{M}_{\mathfrak{X}, \mathfrak{m}}$. We also let \mathfrak{d} denote a compatible metric on \mathfrak{X} . A right norm- (resp. pc-, cc-, rc-) approximate unit for \mathcal{M} is a net $(u_\lambda)_\lambda$ in \mathcal{M} such that $au_\lambda \rightarrow a$ in the norm topology (resp. pct, cct, rct) for every $a \in A$. If $\sup_\lambda \|u_\lambda\|_\infty < \infty$ then $(u_\lambda)_\lambda$ is called bounded. (Bounded) left and two-sided norm- (resp. pc-, cc-, rc-) approximate units are defined similarly. It is clear that every norm-approximate unit is a pc-approximate unit. Suppose that $x \in \mathfrak{X}$ and $\delta > 0$. Throughout, $\mathcal{O}_{x, \delta}$ denotes an open set with $\overline{\mathcal{B}}_{x, \delta} \subseteq \mathcal{O}_{x, \delta} \subseteq \mathcal{B}_{x, 2\delta}$ and $\mathfrak{m}(\mathcal{O}_{x, \delta} \setminus \mathcal{B}_{x, \delta}) < \delta \mathfrak{m}(\mathcal{B}_{x, \delta})$; also $\mathcal{E}_{x, \delta}$ denotes a continuous function from \mathfrak{X} to the interval $[0, 1]$ such that the restriction of $\mathcal{E}_{x, \delta}$ to $\overline{\mathcal{B}}_{x, \delta}$ (resp. $\mathfrak{X} \setminus \mathcal{O}_{x, \delta}$) takes the constant value 1 (resp. 0).

Theorem 3.1. *There is a net in \mathcal{M} which is mutually a right cc-approximate unit and a left rc-approximate unit. Thus the same net is also a two-sided pc-approximate unit.*

Proof. The set of all pairs (S, ϵ) , in which S is a finite subset of \mathfrak{X} and $\epsilon > 0$, with the ordering $((S, \epsilon) \leq (S', \epsilon')) \Leftrightarrow (S \subseteq S', \epsilon' \leq \epsilon)$, becomes a directed set. For any pair (S, ϵ) choose $\delta > 0$ such that $\delta < \epsilon$ and $\mathcal{B}_{y, 2\delta} \cap \mathcal{B}_{y', 2\delta} = \emptyset$ for $y, y' \in S$ with $y \neq y'$, and let $u_{S, \epsilon} = \sum_{y \in S} \frac{1}{\mathfrak{m}(\mathcal{B}_{y, \delta})} \mathcal{E}_{y, \delta} \otimes \mathcal{E}_{y, \delta}$. We show that $(u_{S, \epsilon})_{(S, \epsilon)}$ is the desired net. Let $f \in \mathcal{M}$ and $r > 0$ be arbitrary. Choose $\epsilon > 0$ with $\epsilon < r$ such that for every z, z', x if $\mathfrak{d}(z, z') < \epsilon$ then $|f(x, z) - f(x, z')| < r$. If x is arbitrary then for any pair (S, ϵ) with $y \in S$ we have

$$\begin{aligned} |f \star u_{S, \epsilon} - f|(x, y) &= \frac{1}{\mathfrak{m}(\mathcal{B}_{y, \delta})} \left| \int_{\mathcal{B}_{y, \delta}} [f(x, z) - f(x, y)] \mathfrak{d}\mathfrak{m}(z) + \int_{\mathcal{O}_{x, \delta} \setminus \mathcal{B}_{x, \delta}} f(x, z) \mathcal{E}_{y, \delta}(z) \mathfrak{d}\mathfrak{m}(z) \right| \\ &\leq r + r \|f\|_\infty. \end{aligned}$$

This shows that $f \star u_{S, \epsilon} \rightarrow f$ in cct. Similarly it is proved that $u_{S, \epsilon} \star f \rightarrow f$ in rct. \square

Remark 3.2. *The existence of a right (or left) pc-approximate unit for \mathcal{M} implies that $\text{Spm} = \mathfrak{X}$. An easy proof is as follows. Let $(u_\lambda)_\lambda$ be a right pc-approximate unit. Let U be an arbitrary nonempty open set in \mathfrak{X} and let $f \in \mathbf{C}(\mathfrak{X})$ be such that $f(\mathfrak{X} \setminus U) = \{0\}$ and $f(x) = 1$ for some $x \in U$. Then we have $1 = (1_\mathfrak{X} \otimes f)(x, x) = \lim_\lambda [(1_\mathfrak{X} \otimes f) \star u_\lambda](x, x) = \lim_\lambda \int_U f(z) u_\lambda(z, x) \mathfrak{d}\mathfrak{m}(z)$. This implies that $\mathfrak{m}(U) \neq 0$. Hence $\text{Spm} = \mathfrak{X}$.*

Proposition 3.3. *If \mathcal{M} has a bounded right (or left) pc-approximate unit then \mathfrak{X} is finite.*

Proof. Let $(u_\lambda)_\lambda$ be a right pc-approximate unit for \mathcal{M} bounded by $M > 0$. First of all we show that $\mathfrak{m}(\{x\}) \neq 0$ for every x . Assume, to get a contradiction, that $\mathfrak{m}(\{x\}) = 0$ for some x . Let $\epsilon > 0$ be such that $\epsilon M < 1/2$. There is an open neighborhood U of x with $\mathfrak{m}(U) < \epsilon$. Let $f : \mathfrak{X} \rightarrow [0, 1]$ be a continuous function with $f(x) = 1$ and $f(\mathfrak{X} \setminus U) = \{0\}$. For every λ we have $|(1_\mathfrak{X} \otimes f) \star u_\lambda|(x, x) \leq \int_U |f(z) u_\lambda(z, x)| \mathfrak{d}\mathfrak{m}(z) \leq \epsilon M < 1/2$. But this is impossible because $[(1_\mathfrak{X} \otimes f) \star u_\lambda](x, x) \rightarrow 1$.

Now, since $\mathfrak{m}(\mathfrak{X}) = 1$, it is concluded that \mathfrak{X} must be a countable space. Suppose that \mathfrak{X} is not finite. Then there is an infinite discrete subset $\{x_1, x_2, \dots\}$ of \mathfrak{X} . For every n let $f_n \in \mathcal{M}$ be defined by $f_n(z, z') = 1$ if $z = z' = x_n$ and otherwise $f_n(z, z') = 0$. Then we have $1 = f_n(x_n, x_n) = \lim_{\lambda} (f_n \star u_{\lambda})(x_n, x_n) = \lim_{\lambda} \mathfrak{m}\{x_n\} u_{\lambda}(x_n, x_n)$. It follows that $\mathfrak{m}\{x_n\} \geq 1/M$. But this contradicts $\lim_{n \rightarrow \infty} \mathfrak{m}\{x_n\} = 0$. Hence, \mathfrak{X} is finite. \square

Theorem 3.4. *The following statements are equivalent.*

- (a) \mathfrak{X} is finite.
- (b) \mathcal{M} has a bounded right (or left) pc-approximate unit.
- (c) \mathcal{M} has a unit.

Proof. (b) \Rightarrow (a) is the statement of Proposition 3.3. (c) \Rightarrow (b) is trivial. (a) \Rightarrow (c) is easily verified by analogy with ordinary matrix algebras. \square

Lemma 3.5. *Let $x \in X$. The function $r \mapsto \mathfrak{m}(\mathcal{B}_{x,r})$ is continuous at $r_0 \in [0, \infty)$ if and only if $\mathfrak{m}\{y : \mathfrak{d}(x, y) = r_0\} = 0$. (Note that $\mathcal{B}_{x,0} = \emptyset$.)*

Proof. Straightforward. \square

Lemma 3.6. *The function $x \mapsto \mathfrak{m}(\mathcal{B}_{x,r})$ is continuous at x_0 if $\mathfrak{m}\{y : \mathfrak{d}(x_0, y) = r\} = 0$.*

Proof. For $\epsilon > 0$ by Lemma 3.5 there is $\delta > 0$ such that $\mathfrak{m}(\mathcal{B}_{x_0, r+\delta} \setminus \mathcal{B}_{x_0, r-\delta}) < \epsilon$. Suppose that $y \in \mathcal{B}_{x_0, \delta}$. Then $\mathfrak{m}(\mathcal{B}_{x_0, r-\delta}) \leq \mathfrak{m}(\mathcal{B}_{y, r}) \leq \mathfrak{m}(\mathcal{B}_{x_0, r+\delta})$. So $|\mathfrak{m}(\mathcal{B}_{x_0, r}) - \mathfrak{m}(\mathcal{B}_{y, r})| < \epsilon$. \square

Lemma 3.7. *Let $\delta > 0$ be such that $\mathfrak{m}\{y : \mathfrak{d}(x, y) = \delta\} = 0$ for every $x \in \mathfrak{X}$. Then there exists δ' with $\delta < \delta' < 2\delta$ such that $\mathfrak{m}(\mathcal{B}_{x, \delta'} \setminus \mathcal{B}_{x, \delta}) < \delta \mathfrak{m}(\mathcal{B}_{x, \delta})$ for every $x \in \mathfrak{X}$.*

Proof. Assume, to reach a contradiction, that there is no δ' with the desired properties. For sufficiently large n we have $\delta + n^{-1} < 2\delta$ and hence there is a x_n such that $\mathfrak{m}(\mathcal{B}_{x_n, \delta+n^{-1}}) - \mathfrak{m}(\mathcal{B}_{x_n, \delta}) \geq \delta \mathfrak{m}(\mathcal{B}_{x_n, \delta})$. Without loss of generality we can suppose that the sequence $(x_n)_n$ converges to an element x . Let $r > 0$ be arbitrary. For sufficiently large n we have $\mathfrak{m}(\mathcal{B}_{x_n, \delta+n^{-1}}) \leq \mathfrak{m}(\mathcal{B}_{x, \delta+r})$ and hence $\delta \mathfrak{m}(\mathcal{B}_{x_n, \delta}) \leq \mathfrak{m}(\mathcal{B}_{x, \delta+r}) - \mathfrak{m}(\mathcal{B}_{x_n, \delta})$. It follows from Lemma 3.6 that $\delta \mathfrak{m}(\mathcal{B}_{x, \delta}) \leq \mathfrak{m}(\mathcal{B}_{x, \delta+r} \setminus \mathcal{B}_{x, \delta})$. Letting $r \rightarrow 0$ and using Lemma 3.5 we conclude that $\mathfrak{m}(\mathcal{B}_{x, \delta}) = 0$, a contradiction. \square

Theorem 3.8. *Suppose that the following condition is satisfied. (C1) \mathfrak{X} has a compatible metric \mathfrak{d} under which there is a decreasing sequence $(\delta_n)_n$ of strictly positive numbers such that $\inf_n \delta_n = 0$ and $\mathfrak{m}\{y : \mathfrak{d}(x, y) = \delta_n\} = 0$ for every n and every $x \in \mathfrak{X}$. Then \mathcal{M} has a right (resp. left) norm-approximate unit. Moreover, that approximate unit can be chosen so as to be a sequence.*

Proof. For every n let δ'_n be such that the statement of Lemma 3.7 is satisfied with δ, δ' replaced by δ_n, δ'_n . Let $K_n = \{(x, y) : \mathfrak{d}(x, y) \leq \delta_n\}$ and $U_n = \{(x, y) : \mathfrak{d}(x, y) < \delta'_n\}$. Choose a continuous function $E_n : \mathfrak{X}^2 \rightarrow [0, 1]$ such that $E_n(K_n) = \{1\}$ and $E_n(\mathfrak{X}^2 \setminus U_n) = \{0\}$ and let \mathcal{E}_n (resp. \mathcal{E}'_n) be defined by $(x, y) \mapsto E_n(x, y)/\mathfrak{m}(\mathcal{B}_{y, \delta_n})$ (resp. $(x, y) \mapsto E_n(x, y)/\mathfrak{m}(\mathcal{B}_{x, \delta_n})$). (Note that by Lemma 3.6, $\mathcal{E}_n, \mathcal{E}'_n \in \mathcal{M}$.) Using Lemma 3.7, it is easily verified that $(\mathcal{E}_n)_n$ (resp. $(\mathcal{E}'_n)_n$) is a right (resp. left) norm-approximate unit for \mathcal{M} . \square

Theorem 3.9. *Suppose that the following condition is satisfied. (C2) \mathfrak{X} has a compatible metric \mathfrak{d} under which there exists a sequence $(\delta_n)_n$ satisfying all properties stated in (C1) and, in addition, $\mathfrak{m}(\mathcal{B}_{x,\delta_n}) = \mathfrak{m}(\mathcal{B}_{y,\delta_n})$ for every n and every $x, y \in \mathfrak{X}$. Then \mathcal{M} has a two-sided norm-approximate unit.*

Proof. It is concluded from $\mathcal{E}_n = \mathcal{E}'_n$ where $\mathcal{E}_n, \mathcal{E}'_n$ are as in the proof of Theorem 3.8. \square

Example 3.10. *If \mathfrak{X} is the closure of a nonempty bounded open subset of \mathbb{R}^n with the normalized n -dimensional Lebesgue measure and with the Euclidean metric, then \mathfrak{X} satisfies conditions of Theorem 3.8. More generally, if an open subset of a Riemannian manifold has compact closure \mathfrak{X} then \mathfrak{X} , with the geodesic distance \mathfrak{d} and normalized Riemannian volume \mathfrak{m} , satisfies conditions of Theorem 3.8. Indeed, $\mathfrak{m}\{y : \mathfrak{d}(x, y) = r\} = 0$ for every r and x .*

Example 3.11. *Any closed Riemannian manifold \mathfrak{X} which has constant (positive) sectional curvature (e.g. standard spheres and tori, compact Lie groups with invariant Riemannian metrics), with geodesic distance \mathfrak{d} and normalized Riemannian volume \mathfrak{m} , satisfies conditions of Theorem 3.9. Indeed, in addition to the property mentioned in Example 3.10, we have $\mathfrak{m}(\mathcal{B}_{x,r}) = \mathfrak{m}(\mathcal{B}_{y,r})$ for every r, x, y .*

Example 3.12. *Let \mathfrak{X} be a second countable compact Hausdorff group. It is well-known that \mathfrak{X} has a compatible bi-invariant metric \mathfrak{d} i.e. $\mathfrak{d}(zxz', zyz') = \mathfrak{d}(x, y)$ for every $x, y, z, z' \in \mathfrak{X}$ (see [8] or [6, Corollary A4.19]). We show that \mathfrak{d} with the normalized Haar measure \mathfrak{m} satisfies (C1) and hence (because of invariant property of \mathfrak{m}) satisfies (C2): Suppose, on the contrary, that there is no sequence $(\delta_n)_n$ satisfying (C1) for \mathfrak{d} . So there must be $\epsilon > 0$ such that $\mathfrak{m}\{y : \mathfrak{d}(e, y) = r\} \neq 0$ for every nonzero $r < \epsilon$; thus $\mathfrak{m}(\mathcal{B}_{e,\epsilon}) = \infty$, a contradiction.*

4. THE CENTER OF \mathcal{M}

It is clear that if \mathfrak{X} is finite then the center of \mathcal{M} is the one-dimensional subalgebra of scalar multiples of the unit of \mathcal{M} . But in the infinite case the situation is different:

Theorem 4.1. *If \mathfrak{X} is infinite then the center of \mathcal{M} is zero.*

Proof. Suppose that f is in the center of \mathcal{M} . Let x, y be arbitrary in \mathfrak{X} with $x \neq y$, and $\delta > 0$ be such that $\mathfrak{d}(x, y) > 4\delta$. Let $g := \frac{1}{\mathfrak{m}(\mathcal{B}_{x,\delta})} \mathcal{E}_{x,\delta} \otimes \mathcal{E}_{x,\delta}$ and $h_\delta := \frac{1}{\mathfrak{m}(\mathcal{B}_{x,\delta})} \mathcal{E}_{x,\delta} \otimes \mathcal{E}_{y,\delta}$. Then we have $f \star g(x, y) = 0$ and hence $g \star f(x, y) = 0$. We have,

$$|f|(x, y) = |g \star f - f|(x, y) \leq \frac{1}{\mathfrak{m}(\mathcal{B}_{x,\delta})} \int_{\mathcal{B}_{x,\delta}} |f(z, y) - f(x, y)| d\mathfrak{m}(z) + \delta \|f\|_\infty.$$

By this inequality and continuity of f we conclude that $f(x, y) = 0$. Also, a simple computation shows that $\lim_{\delta \rightarrow 0} f \star h_\delta(x, y) = f(x, x)$ and $\lim_{\delta \rightarrow 0} h_\delta \star f(x, y) = f(y, y)$. Thus we have $f(x, x) = f(y, y)$. Now, suppose that \mathfrak{X} is infinite. Then there is a sequence $(x_n)_{n \geq 0}$ such that $x_n \rightarrow x_0$ and $x_0 \neq x_n$ for every $n \geq 1$. Thus $f(x_0, x_0) = \lim_{n \rightarrow \infty} f(x_0, x_n) = 0$ and hence $f(x, x) = f(x_0, x_0) = 0$. This completes the proof. \square

5. THE IDEAL STRUCTURE OF \mathcal{M}

It is clear that the involution $*$ induces a one-to-one correspondence between norm- (resp. rc-, cc-, pc-) closed right ideals and norm- (resp. cc-, rc-, pc-) closed left ideals of \mathcal{M} . Also any self-adjoint right or left ideal is a two-sided ideal. The rc-closure of any right ideal is a right ideal and the cc-closure of any left ideal is a left ideal. For any norm-closed linear subspace V of $\mathbf{C}(\mathfrak{X})$ we let $\mathcal{R}_V := \{f \in \mathcal{M} : f(\cdot, y) \in V\}$ and $\mathcal{L}_V := \{f \in \mathcal{M} : f(x, \cdot) \in V\}$. It is clear that $\mathcal{R}_V^* = \mathcal{L}_{\bar{V}}$ and $\mathcal{L}_V^* = \mathcal{R}_{\bar{V}}$ where $\bar{V} := \{\bar{f} : f \in V\}$.

Theorem 5.1. \mathcal{R}_V (resp. \mathcal{L}_V) is a cc-closed right (resp. rc-closed left) ideal in \mathcal{M} . Moreover, if V is pc-closed then \mathcal{R}_V (resp. \mathcal{L}_V) is pc-closed.

Proof. It is clear that \mathcal{R}_V is a cc-closed linear subspace of \mathcal{M} . Let $f \in \mathcal{R}_V$ and $g \in \mathcal{M}$. For every y let $h_y : \mathfrak{X} \rightarrow V$ be defined by $h_y(z) = f(\cdot, z)g(z, y)$. Then the Bochner integral $\int_{\mathfrak{X}} h_y d\mathbf{m}$ exists and belongs to V ([7, Proposition 1.31]). Since $f \star g(\cdot, y) = \int_{\mathfrak{X}} h_y d\mathbf{m}$, we have $f \star g \in \mathcal{R}_V$. Thus \mathcal{R}_V is a right ideal. Also, $\mathcal{L}_V = \mathcal{R}_V^*$ is a rc-closed left ideal. The second part of the theorem is trivial. \square

Theorem 5.2. Let R be a norm-closed right ideal of \mathcal{M} and let $V = \{f(\cdot, y) : f \in R, y \in \mathfrak{X}\}$. Then V is a norm-closed linear subspace of $\mathbf{C}(\mathfrak{X})$ and the cc-closure of R is equal to \mathcal{R}_V . Moreover, if R is pc-closed then V is pc-closed.

Proof. Suppose that $f \in R$ and $y \in \mathfrak{X}$. Let $\epsilon > 0$ be arbitrary and $\delta > 0$ with $\delta < \epsilon$ be such that if $\mathfrak{d}(z, z') < \delta$ then $|f(x, z) - f(x, z')| < \epsilon$ for every x . Then for every x, y' we have $|\frac{1}{\mathfrak{m}(\mathcal{B}_{y, \delta})}[f \star (\mathcal{E}_{y, \delta} \otimes 1)](x, y') - f(x, y)| \leq \epsilon + \epsilon \|f\|_{\infty}$. This implies that there exists $F_{f, y} \in R$ with $F_{f, y}(x, z) = f(x, y)$ for every x, z . Let $h, h' \in V$. Let $f, f' \in R$ and $y, y' \in \mathfrak{X}$ be such that $h = f(\cdot, y)$ and $h' = f'(\cdot, y')$. We have $h + h' = [F_{f, y} + F_{f', y'}](\cdot, z)$ for any arbitrary z and thus $h + h' \in V$. This shows that V is a linear subspace. Suppose that $g \in \mathbf{C}(\mathfrak{X})$ is a limit point of V . There are sequences $(f_n)_n \in R$ and $(y_n)_n \in \mathfrak{X}$ such that $f_n(\cdot, y_n) \rightarrow g$. It is clear that the sequence $(F_{f_n, y_n})_n \in R$ converges to an element G of R with $G(\cdot, z) = g$ for every z . This shows that V is norm-closed. (A similar argument shows that if R is pc-closed then V is pc-closed.) To complete the proof, it is enough to show that if $K \in \mathcal{R}_V$ then there exists a net in R converging to K in cct. Let $K \in \mathcal{R}_V$ be fixed. For every y there are $k_y \in R$ and $\alpha(y) \in \mathfrak{X}$ such that $K(\cdot, y) = k_y(\cdot, \alpha(y))$. For every $\epsilon > 0$ and every finite subset S of \mathfrak{X} there exists $\delta > 0$ with the following three properties.

- $\delta \|k_y\|_{\infty} < \epsilon/2$ for every $y \in S$.
- $\mathcal{B}_{y, 2\delta} \cap \mathcal{B}_{y', 2\delta} = \emptyset$ for $y, y' \in S$ with $y \neq y'$.
- If $\mathfrak{d}(z, z') < 2\delta$ then $|k_y(x, z) - k_y(x, z')| < \epsilon/2$ for every $y \in S$.

Let $K_{S, \epsilon} := \sum_{y \in S} \frac{1}{\mathfrak{m}(\mathcal{B}_{\alpha(y), \delta})} h_y \star (\mathcal{E}_{\alpha(y), \delta} \otimes \mathcal{E}_{y, \delta}) \in R$. Then $\|K_{S, \epsilon}(\cdot, y) - G(\cdot, y)\|_{\infty} < \epsilon$ for every $y \in S$. Considering the set of all pairs (S, ϵ) as a directed set in the obvious way, shows that $K_{S, \epsilon} \xrightarrow{\text{cct}} K$. \square

Passing through the involution and using Theorem 5.2, we conclude that for any norm-closed left ideal L of \mathcal{M} , $V := \{f(x, \cdot) : f \in L, x \in \mathfrak{X}\}$ is a norm-closed linear subspace and rc-closure of L is equal to \mathcal{L}_V . Moreover, if L is pc-closed then V is pc-closed.

Corollary 5.3. *The mapping $V \mapsto \mathcal{R}_V$ (resp. $V \mapsto \mathcal{L}_V$) establishes a 1-1 correspondence between norm-closed linear subspaces of $\mathbf{C}(\mathfrak{X})$ and cc-closed right (resp. rc-closed left) ideals of \mathcal{M} , and also between pc-closed linear subspaces of $\mathbf{C}(\mathfrak{X})$ and pc-closed right (resp. left) ideals of \mathcal{M} . In particular, 1-dimensional and norm-closed 1-codimensional subspaces of $\mathbf{C}(\mathfrak{X})$ correspond respectively to minimal and maximal cc-closed right (resp. rc-closed left) ideals of \mathcal{M} .*

Corollary 5.4. *There is no nontrivial ideal in \mathcal{M} mutually closed under both cct and rct. In particular, there is no nontrivial pc-closed ideal in \mathcal{M} .*

Proof. Let I be a nonzero cc-closed and rc-closed ideal. There are closed linear subspaces $V, W \subseteq \mathbf{C}(\mathfrak{X})$ such that $I = \mathcal{R}_V = \mathcal{L}_W$. Since $V \neq 0$ there are $f_0 \in V$ and $x_0 \in \mathfrak{X}$ with $f_0(x_0) = 1$. For every $g \in \mathbf{C}(\mathfrak{X})$ we have $f_0 \otimes g \in \mathcal{R}_V$. Thus $g = (f_0 \otimes g)(x_0, -) \in W$ and $W = \mathbf{C}(\mathfrak{X})$. So, $I = \mathcal{M}$. \square

6. CANONICAL REPRESENTATIONS OF \mathcal{M}

For a Banach algebra A a Banach space E is called Banach left A -module if E is a left A -module in the algebraic sense and such that the action of A on E is a bounded bilinear operator. Banach right A -modules and Banach A -bimodules are defined similarly. Let $\mathbf{B}(E)$ denote the Banach algebra of bounded linear operators on E and $\mathbf{K}(E) \subseteq \mathbf{B}(E)$ be the closed ideal of compact operators. Any Banach left A -module structure on E gives rise to a bounded representation $A \rightarrow \mathbf{B}(E)$, $a \mapsto (\omega \mapsto a\omega)$, and vice versa. The statements of the following theorem are standard results and can be find for instance in [5].

Theorem 6.1. *Let E denote any of the Banach spaces $\mathbf{L}^p(\mathfrak{m})$ ($1 \leq p \leq \infty$) or $\mathbf{C}(\mathfrak{X})$. Then $\rho : \mathcal{M} \rightarrow \mathbf{K}(E)$, defined by $[\rho(f)g](x) = \int_{\mathfrak{X}} f(x, y)g(y)d\mathfrak{m}(y)$ ($g \in E$), is a well-defined faithful bounded representation. Moreover, the following statements hold.*

- (i) *In the case that $E = \mathbf{L}^2(\mathfrak{m})$, ρ is a $*$ -representation.*
- (ii) *In the case that $E = \mathbf{L}^\infty(\mathfrak{m})$ or $E = \mathbf{C}(\mathfrak{X})$, ρ is isometric.*

It is clear that for any Banach space E , $\mathbf{C}(\mathfrak{X}; E)$ is a Banach right (resp. left) \mathcal{M} -module in the canonical way. Its module action is denoted by the same symbol \star and is given by $(g \star f)(y) = \int_{\mathfrak{X}} g(z)f(z, y)d\mathfrak{m}(z)$ (resp. $(f \star g)(x) = \int_{\mathfrak{X}} f(x, z)g(z)d\mathfrak{m}(z)$) for $f \in \mathcal{M}$ and $g \in \mathbf{C}(\mathfrak{X}; E)$. Similarly, $\mathbf{C}(\mathfrak{X}^2; E)$ becomes a Banach \mathcal{M} -bimodule.

7. DERIVATIONS ON \mathcal{M}

Let A be a Banach algebra and E be a Banach A -bimodule. A (bounded) *derivation* from A to E is a (bounded) linear map $D : A \rightarrow E$ satisfying $D(ab) = aD(b) + D(a)b$ ($a, b \in A$). D is called *inner* if there exists $\omega \in E$ such that $D(a) = a\omega - \omega a$ for every a . D is called *approximately inner* [4] if there is a net $(\omega_\lambda)_\lambda$ in E such that $D(a) = \lim_\lambda a\omega_\lambda - \omega_\lambda a$. If $(\omega_\lambda)_\lambda$ can be chosen so as to be a sequence then D is called *sequentially approximate inner*.

Theorem 7.1. *Suppose that the condition (C2) of Theorem 3.9 is satisfied, and let E be a Banach \mathcal{M} -bimodule such that its module operation $\diamond : \mathcal{M} \otimes E \otimes \mathcal{M} \rightarrow E$ is continuous w.r.t. injective tensor norm, and such that for every norm approximate unit $(\mathcal{E}_n)_n$ of \mathcal{M} we*

have $\mathcal{E}_n \diamond \omega \rightarrow \omega$ for every $\omega \in E$. Then any bounded derivation from \mathcal{M} to E is sequentially approximate inner.

Proof. Let $D : \mathcal{M} \rightarrow E$ be a bounded derivation. Let $\Gamma : \mathcal{M} \check{\otimes} \mathcal{M} \rightarrow E$ be the bounded linear map defined by $f \otimes g \mapsto f \diamond D(g)$. Also let $\Lambda : \mathcal{M} \check{\otimes} \mathcal{M} \rightarrow \mathcal{M}$ denote the convolution product. It is not hard to verify the following two identities for $h \in \mathcal{M}$ and $F \in \mathcal{M} \check{\otimes} \mathcal{M}$.

$$\Gamma(h \star F) = h \diamond \Gamma(F), \quad \Gamma(F \star h) = \Lambda(F) \diamond D(h) + \Gamma(F) \diamond h.$$

Let the sequence $(\delta_n)_n$ be as in the statement of Theorem 3.9 and let $\alpha_n = \mathbf{m}(\mathcal{B}_{x, \delta_n})$ for every $x \in \mathfrak{X}$. By Lemma 3.7 there is r_n such that $\delta_n < r_n < 2\delta_n$ and $\mathbf{m}(\mathcal{B}_{x, r_n} - \mathcal{B}_{x, \delta_n}) < \delta_n \alpha_n$. Choose a continuous function $G_n : \mathfrak{X}^2 \rightarrow [0, 1]$ such that G_n has constant values 1 and 0 respectively on $\{(x, y) : \mathfrak{d}(x, y) \leq \delta_n\}$ and $\{(x, y) : \mathfrak{d}(x, y) \geq r_n\}$, and let $\mathcal{G}_n \in \mathbf{C}(\mathfrak{X}^4)$ be defined by $\mathcal{G}_n(x, z, z', y) = \frac{1}{\alpha_n} G_n(x, y)$. Note that we have $\Lambda(\mathcal{G}_n) = \frac{1}{\alpha_n} G_n$. It is not hard to verify that $(\Lambda(\mathcal{G}_n))_n$ is a two-sided norm-approximate unit for \mathcal{M} and $\lim_{n \rightarrow \infty} f \star \mathcal{G}_n - \mathcal{G}_n \star f = 0$ for every $f \in \mathcal{M}$. Let $K_n = \Gamma(\mathcal{G}_n) \in E$. For the sequence $(K_n)_n$ and $h \in \mathcal{M}$ we have,

$$\begin{aligned} \lim_{n \rightarrow \infty} h \diamond K_n - K_n \diamond h &= \lim_{n \rightarrow \infty} h \diamond \Gamma(\mathcal{G}_n) - \Gamma(\mathcal{G}_n) \diamond h \\ &= \lim_{n \rightarrow \infty} \Gamma(h \star \mathcal{G}_n) - \Gamma(\mathcal{G}_n \star h) + \Lambda(\mathcal{G}_n) \diamond D(h) \\ &= \Gamma\left(\lim_{n \rightarrow \infty} h \star \mathcal{G}_n - \mathcal{G}_n \star h\right) + D(h) \\ &= D(h). \end{aligned}$$

This completes the proof. \square

For any Banach space E , the Banach \mathcal{M} -bimodule $\mathbf{C}(\mathfrak{X}^2; E)$, mentioned in the preceding section, satisfies the conditions of Theorem 7.1.

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